

Game of Power Allocation on Networks

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Abstract—This paper develops a distributed resource allocation game to study countries’ pursuit of targets such as self-survival in the networked international environment. The paper has two general contributions: firstly, it contributes the basic idea that countries’ behavior, which is power allocation, is a basic human behavior of resource allocation and the development of this game is the first time countries’ behavior has ever been rigorously studied from a resource allocation perspective; secondly, the game itself has an intrinsically interesting and novel mathematical structure — it actually presents a new technical problem with a surprising amount of informative predictions which arise from the rich parameter space that defines all kinds of possibilities for the networked international environments. The predictions both motivate major theoretical results (e.g., Nash equilibrium existence) and shed light on real world politics.

Index Terms—power allocation, survival, friends, foes, networks

I. INTRODUCTION

A. Research Question: The Problem of Survival.

What fundamentally drives politics in international relations? A basic assumption in the general theories of international relations, especially in realism, is that countries primarily seek self-survival as the minimal target, and, when this target is satisfied, would pursue universal domination at a maximum target. It is important to note, however, that a country’s universal domination is also a survival-related target just as its own self-survival, and in particular that it relates to the survival of multiple fronts — universal domination at least reflects in jeopardizing the foes’ survival, and also, invariably, in guaranteeing the allies’ survival. Much like in the inaugural address of John F. Kennedy in 1961, “we shall pay any price, bear any burden, meet any hardship, support any friend, oppose any foe to assure the survival and the success of liberty”. Therefore, it is the fundamental driving force of politics in international relations that countries relentlessly pursue targets which relate to the survival of its multiple fronts — positively to that of itself and allies and negatively to that of its foes.

To pursue any target in international relations, the “currency” is “power” — at any point of time, a country has a certain stock of military power (i.e., any mobilizable form of conventional and unconventional war-making capability, including weaponry, manpower and military expenditure) to pursue any international affair.

Countries always pursue their targets in a “networked international environment”. The *networked international en-*

vironment consists of a collection of countries, each of which has its own total power, external relations (i.e., friends and foes), and distinctive preferences structures (i.e., priorities in terms of fulfilling all its targets, which, as will be mentioned, can be deduced from the affinity/animosity levels with every friend/foe).

At any point of time in such a networked international environment, countries actually engage in a basic resource allocation behavior, i.e., “power allocation”. A country’s *power allocation* means to locate their total power towards their friends and itself as support and towards their foes as threats, in a similar vein as governments allocating budgets, firms allocating staff and equipment, and engineers allocating information flows. Even though countries might gradually increase its power over time, this kind of power allocation happens at any point of time.

Guided by this perspective, this paper focuses on the study of *all* possible directions of power allocation in *any* given environment and their implications (e.g., for countries’ own survival), which can restore many scenarios of real-world politics with a great level of theoretical precision and even predict for unintuitive scenarios. In other words, the research question here is to understand whether, given a networked international environment, a country can fulfill its targets, for instance, self-survival.

B. The Method.

A *distributed resource allocation game on networks* is developed to study countries’ *strategic* behavior of power allocation in the networked international environment. It is worth emphasizing that this is a new technical problem and that the innate mathematical structure of the game enables it to provide a rich set of predictions, where the ability arise from the richness in the parameters of the game which define a myriad of possibilities for the networked international environments. For instance, this game does not impose a specific interaction pattern about the players, as in a commonly known N -player game which usually assumes every player to interact with every other player, which is equivalent to assuming the interactions happen on a *complete graph*. As a contrast, this game can take place on any graph structure, and the graph structure itself is an important mathematical parameter for predictions, which, in particular, also matches with the reality. To the best of my knowledge, it is the first time that a distributed resource allocation game on networks is applied in the context of countries’ behavior¹.

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¹Other examples of distributed resource allocation games on networks include the famous congestion games.

There are even very few papers on similar problems, and an example is [1], which focuses on the problem of whether world trade functions to alleviate the intensities of militarized conflicts.

C. Application Scope.

The application scope lies in two broad aspects below:

- 1) an estimation of the odds in whose favor in any given networked international environment — for example, one particularly meaningful question is: can countries always survive in that environment?
- 2) an understanding of countries’ “everyday politics”, which does not necessarily involve real conflicts but various other forms of power allocation (e.g. army deployment) just in order to deter the foes and reassure the allies, as evident in the case of great powers’ national security agendas such as that of the US and China.

This application scope makes possible the study of a series of scenarios in either the history of international relations or the contemporary real-world decision making.

D. Plan of the Paper

The paper proceeds with two main steps,

- 1) first by rigorously defining the networked international environment by discussing its every parameter.
- 2) second by introducing the power allocation game in this environment as well as informative examples that motivate a list of general predictions, which have both theoretical (e.g., Nash equilibrium existence) and real-world implications.

II. THE NETWORKED INTERNATIONAL ENVIRONMENT

In the context of the power allocation game, the networked international environment can also be termed as “the power allocation environment”. Formally, a networked international environment is defined through rigorously defining the collection of the parameters below.

- 1) the collection of countries
- 2) countries’ relations
- 3) countries’ total power
- 4) countries’ preference structures for their targets, which, as will be discussed, reflect in the preferences for all the possible “power allocation outcomes”²

A. Countries.

$\mathcal{V} = \{1, 2, \dots, n\}$ is the set of labels that represents the collection of countries in the environment, where i denotes country i .

Countries are the primary actors in international affairs and the *autonomous agents in the networked international environment*. The CIA World Factbook defines countries as “a wide variety of dependencies, areas of special sovereignty,

uninhabited islands, and other entities in addition to the traditional countries or independent states”. For instance, the countries, shown in the map of Europe in Figure 1, were involved in WWI as of July 28, 1914. The collection of countries is $\{GMY, UKG, RUS, FRN, AUH, ITA, ROM, SER, NOR\}$, which can be represented by $V = \{1, 2, \dots, 9\}$.



Fig. 1: European Countries Involved in WWI on Jul 28, 1914 ($n = 9$)

B. Relations.

A symmetric matrix $R = [r_{ij}]_{n \times n}$ is the matrix representation of the values of all the relations among countries. A function $r : \mathcal{V} \times \mathcal{V} \rightarrow \{\text{ally}, \text{foe}, \text{none}, \text{self}\}$ returns for countries i and $j \in \mathcal{V}$ a value of their relation, denoted as r_{ij} . If $i \neq j$, $r_{ij} \in \{\text{ally}, \text{foe}, \text{none}\}$, which denotes the value of the bilateral relation between i and j . Otherwise, $r_{ii} = \text{self}$, which denotes country i ’s self relation. Conveniently, the following can be defined:

- 1) The foe relations set is $\Phi = \{\{i, j\} | r_{ij} = \text{foe}\}$, with the foe set of $i \in V$ being $\Phi_i = \{j | r_{ij} = \text{foe}\}$.
- 2) The ally relations set is $A = \{\{i, j\} | r_{ij} = \text{ally}\}$, with the ally set of $i \in V$ being $A_i = \{j | r_{ij} = \text{ally}\}$.

The most basic unit of analysis for relations is dyadic ties. Dyadic ties are usually a mix of *cooperative* or *conflictual* elements, with a cooperative activity in military, political (e.g., bilateral defense cooperation, military-to-military cooperation and homeland security cooperative), or economic forms (e.g., bilateral trade), and a conflictual activity ranging from threats of use of force to militarized disputes.

As unexpected events occur and international environments will change with time, the dyadic ties between two countries can be assessed to be ally, foe, or none (i.e., having no specific relations) by weighing the cooperative against the conflictual elements that suit the particular time instant. This implies that regardless of how the cooperative and conflictual elements are assessed, any relation must be *de facto*. A country dyad should be counted as foes and not allies in a given period in which they are concurrently involved in militarized conflicts/tensions, regardless of any *de jure* defense commitments. An example is that the two NATO members, Turkey and Greece, should be counted as foes instead during the 1995 Imia Crisis.

C. Power.

Let $p = [p_i]_{1 \times n}$, where p_i denotes the total power of country i in \mathcal{V} and $p \in \mathbb{R}_{\geq 0}^n$.

²The definitions of “power allocation strategy profiles”, and “power allocation outcomes” are part of the definition of preferences for the power allocation outcomes. Nevertheless, this paper treats them three as separate parameters.

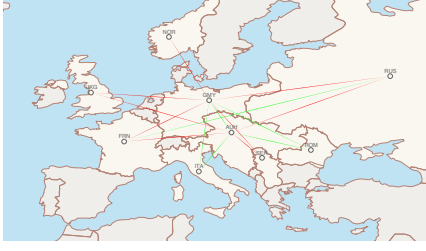


Fig. 2: Red line denotes conflictual status and green line denotes defensive commitment in WWI-era Europe.

In international relations, a country's power can be primarily understood as its military power, backed by its economic strength. Measures of economic strength such as the GDP could be imperfect measures of a country's power in international affairs.

This concept of power is an aggregate measure first developed in [2]—National power is an aggregate measure of all the conventional and nuclear war making capabilities mobilizable for use in international security, being composed of geography, natural resources (e.g., food, raw materials), industrial capacity, military preparedness (e.g., technology, leadership, quantity and quality of armed forces), population (e.g., distribution and trends), national character, national morale and quality of diplomacy and government [2].

The real-world contexts underlying the environment (e.g., the nature of countries' engagement) are crucial to evaluating power. Several important aspects include, for instance, whether the engagement is real conflict or peacetime escalation. In peace time, power can be aggregated with any form of conventional or nuclear war-making capabilities across land, sea and air. For instance, manpower, land systems, air power, naval power, logistical and financial support, etc. all make up a country's military power. In war times, a country's power includes all its *mobilized* resources available for the war. Especially, the national morale would become relevant for a comparison of countries' power — even if two countries have the same amount of power, the mutiny and morale problems could mark the difference between victory and defeat after controlling for the other factors.

D. Preferences for Power Allocation Outcomes

In pursuing targets in an international environment, countries are also driven by their distinctive belief systems. This is equivalent to saying that they may have different priorities in terms of pursuing the targets. Technically, priorities/preferences in terms of pursuing the targets really reflect in preferences in terms of pursuing the possible "power allocation outcomes". Therefore, the following sections first define the set of "power allocation strategy profiles", then the set of "power allocation outcomes" mapped from it, and lastly discuss countries' "preferences regarding the outcomes".

1) *Set of Power Allocation Strategy Profiles*: A function $f : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{W}$ defines the set of power allocation strategy

profiles

$$\mathcal{W} = \{W : W = [w_{ij}]_{n \times n}, W \in \mathbb{R}_{\geq 0}^{n \times n} \text{ and} \\ \|W_{i,\bullet}\|_1 = \sum_{j \in \mathcal{V}} w_{ij} = \sum_{j \in A_i \cup \Phi_i \cup \{i\}} w_{ij} = p_i\}$$

where the equality regarding country i 's own allocation, $W_{i,\bullet}$, is its total power constraint below. Therefore, \mathcal{W} is a convex and compact subset of $\mathbb{R}_{\geq 0}^{n \times n}$.

Power allocation captures two aspects of countries' *everyday politics*. First, countries need to be constantly and simultaneously involved with their multiple contingencies — defending the home land by allocating power to oneself, protecting the allies by allocating power to support them, and offending the foes by allocating power to oppose them.

Second, in everyday politics, the power allocated do not have to be interpreted as blood, toil, tears and sweat in real conflicts. Offenses towards the foes can be interpreted either as real attacks, or mere postures and potentials [3, 4, 5, 6], in which case deterrence strategies are sufficient to deter the foes and reassure the allies. For instance, countries' weaponry systems can be set up for precautionary purposes, such as the anti-ballistic missile system (e.g., THAAD) and the intercontinental ballistic missiles (e.g., ICBM) targeting at foes. Only under extreme circumstances will the weaponry systems be actually put into use. Defense support for allies take numerous forms in reality. In the case of the US defense support for NATO allies to counteract the perceived potentials for conflicts from foes, the US maintains steady force presence in US-European Command, sends a battalion-sized unit from the United States to Europe twice a year for up to two months per rotation, contributes to the European missile defense protecting against emerging threats from outside of the Euro-Atlantic area, and participates in exercises such as in the Baltic Sea and Poland [7].

2) *Set of Power Allocation Outcome*: A power allocation outcome includes the *basic state* of every country in the environment defined through comparing their *total support* to *total threats*.

- *Total Support*. A function $\xi_i : \mathcal{W} \rightarrow \mathbb{R}_{\geq 0}$ defines country i 's total support $\xi_i(W)$:

$$\xi_i(W) = w_{ii} + \sum_{j \in A_i} w_{ji} + \sum_{j \in \Phi_i} \min\{w_{ij}, w_{ji}\}$$

where w_{ii} is i 's self-defense, $\sum_{j \in A_i} w_{ji}$ is i 's received total defense support and $\sum_{j \in \Phi_i} \min\{w_{ij}, w_{ji}\}$ is i 's total *effective defenses*.

- *Total Threats*. A function $\theta_i : \mathcal{W} \rightarrow \mathbb{R}_{\geq 0}$ defines country i 's total threats $\theta_i(W)$ ³:

³ $\sum_{j \in \Phi_i} \min\{w_{ij}, w_{ji}\}$ is a normalization, and, given the way $s_i(W)$ is determined by comparing $\xi_i(W)$ to $\theta_i(W)$, can be alternatively $\sum_{j \in \Phi_i} w_{ij}$, with none of the results in the paper being affected.

$$\theta_i(W) = \sum_{j \in \Phi_i} w_{ji}$$

- *Basic State*. A function $s_i : \mathcal{W} \rightarrow \{\text{safe}, \text{unsafe}\}$ defines country i 's basic state⁴.

$$s_i(W) = \begin{cases} \text{safe}, & \text{if } \xi_i(W) \geq \theta_i(W) \\ \text{unsafe}, & \text{if } \xi_i(W) \leq \theta_i(W) \end{cases}$$

In particular, if $\xi_i(W) = \theta_i(W)$, $s_i(W) = \text{precarious}$.

Set of Power Allocation Outcomes: A basic state vector, $s(W) = [s_i(W)]_{1 \times n}$, formally represents a *power allocation outcome* mapped from a power allocation strategy profile W . Conveniently, let the *set of all possible power allocation outcomes* be

$$\mathcal{S} = \{s^i, i \in \{1, 2, \dots, 2^N\}\}$$

where, since each country's basic state takes one possibility of two values, its cardinality is obviously 2^N .

3) *Preferences for Power Allocation Outcomes*: Let $(\mathcal{S}, \preccurlyeq_i)$ be country i 's *preference order* regarding the 2^N possible power allocation outcomes in \mathcal{S} , where \preccurlyeq_i denotes weak preference (i.e., strictly preferring or being indifferent). The preference order of the outcomes actually reflect a country's preference order of its targets. For instance, a country with two allies might need to compare two targets (and also power allocation outcomes), each of which has only one ally being *safe* and the other being *unsafe*, especially when it has to make a choice between them.

Despite that different countries may hold different preference orders regarding the same set of outcomes, the most intuitive and universal aspects to *any* country's preference order are summarized by the choice axioms below.

Choice Axioms for $(\mathcal{S}, \preccurlyeq_i)$. All others being equal, the axioms first involve the comparison of two outcomes $s(W)$ and $s(\hat{W})$ along any single dimension (i.e., the basic states of individual countries), then the comparison of outcomes themselves, and lastly the most basic assumption in international relations theory — *the first priority of self-survival*.

- *Basic State Comparison*. $s_j(W) \preccurlyeq_i s_j(\hat{W})$ ⁵ if

- 1) $r_{ij} = \text{ally or self}$,
 - a) $\xi_j(W) \geq \theta_j(W)$, or
 - b) $\xi_j(W) < \theta_j(W)$ ⁶.
- 2) $r_{ij} = \text{foe}$,

⁴The terminologies used for basic states are only for representation convenience and their interpretations should always be based on $\xi_i(W)$ and $\theta_i(W)$.

⁵Obviously, this incorporates the assumptions that when $r_{ij} = \text{ally}$ or self , $s_j(W) \preccurlyeq_i s_j(\hat{W})$ if $\xi_j(\hat{W}) > \theta_j(\hat{W})$ and $\xi_j(W) \leq \theta_j(\hat{W})$ or $\xi_j(\hat{W}) = \theta_j(\hat{W})$ and $\xi_j(W) < \theta_j(\hat{W})$, and that when $r_{ij} = \text{foe}$, $s_j(W) \preccurlyeq_i s_j(\hat{W})$ if $\xi_j(\hat{W}) < \theta_j(\hat{W})$ and $\xi_j(W) \geq \theta_j(\hat{W})$ or $\xi_j(\hat{W}) = \theta_j(\hat{W})$ and $\xi_j(W) > \theta_j(\hat{W})$.

⁶ $s_j(W) \neq \text{safe}$

- a) $\xi_j(W) \leq \theta_j(W)$, or
- b) $\xi_j(W) > \theta_j(W)$ ⁷.
- 3) $r_{ij} = \text{none}$,
here $s_j(W) \sim_i s_j(\hat{W})$.

- *Basic State Vector Comparison*. $s(W) \preccurlyeq_i s(\hat{W})$ if $s_j(W) \preccurlyeq_i s_j(\hat{W})$, $j \in V$.
- *First Priority of Self-Survival*. $s(W) \prec_i s(\hat{W})$ if $\xi_i(W) < \theta_i(W)$, and $\xi_i(\hat{W}) \geq \theta_i(\hat{W})$.

Since the axioms only capture a few intuitive aspects such as a country always tries to jeopardize the survival of the foes and maintain the survival of itself and its allies, they make $(\mathcal{S}, \preccurlyeq_i)$ a *partial order* — the aforementioned example of two targets cannot be compared under this partial order.

However, as each axiom is mutually exclusive, a *total order* that satisfies these axioms must exist. Also, as discussed in the Appendix of Chapter 2 (“Theory of Power Allocation”) of [8], a level of affinity/animosity attached to each relation would extend the partial order into a total order, making all outcomes comparable. This assumption abstracts for the varying importances attached to the cooperative or conflictual relations in reality, for which ideological differences, economic dependence and voting patterns in UN can be empirical manifestations. The determining factors could be highly complex, spanning from the simplest geographic proximity among countries to their historical, economic and cultural origins.

E. Graphical Representation

A directed and connected⁸ graph on N vertices and $2M$ edges, $\mathbb{G} = (\mathcal{V}, \mathcal{E})$, is a convenient representation of the environment with N countries and M bilateral relations ($N, M \in \mathbb{Z}_+$).

- 1) \mathbb{G} 's vertex set $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ is also the country set \mathcal{V} . The vertex set \mathcal{V} also represents all the *self* relations.
- 2) The edge set $\mathcal{E} = \{e_1, e_2, \dots, e_{2M}\}$ represents M bilateral relations. Each element in \mathcal{E} represents a directed edge attached to a bilateral relation — in \mathcal{E} , the ordered pairs (i, j) and (j, i) represents the two edges with opposite directions for the relation $\{i, j\}$.
- 3) i 's neighbors on the graph are either its allies or foes. The weighted version of the above graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ can represent a power allocation strategy profile.

- 1) w_{ij} , which represents i 's investments on the bilateral relation r_{ij} , is drawn as the edge weight of (i, j) .
- 2) w_{ii} , which represents i 's self-defense, is drawn as the vertex weight of i .
- 3) Prior to power allocation, total power c_i is trivially the vertex weight of i .

F. Summary

To summarize, a networked international environment can be specified as the collection of all the parameters defined previously, $\tau = \{\mathcal{V}, R, p, \mathcal{W}, \xi_i, \theta_i, \mathcal{S}, \preccurlyeq_i\}$.

⁷ $s_j(W) \neq \text{unsafe}$

⁸The connected graph is assumed only for simplicity. For unconnected graphs, we only need to analyze the connected subgraphs separately.

III. POWER ALLOCATION GAME

The power allocation game takes place in a networked international environment, or on the graph that represents a given relation configuration R . Assuming the networked international environment is equivalent to assuming that the game includes all these elements of τ , the definition of the networked international environment conveniently provides the formal description of the power allocation game $\Gamma = \{\mathcal{V}, R, p, \mathcal{W}, \xi_i, \theta_i, \mathcal{S}, \preceq_i\}$.

A. Game Rules

The power allocation game is *static*, where each country's strategy is to allocate its fixed amount of total power towards its own relations subject to its preference for all possible power allocation outcomes.

A *complete information* framework is assumed, where countries have full knowledge of the environment, and this framework is also suitable for studying the scenarios where countries act upon individual and subjective perceptions of the environment.

B. Equilibrium Concept

Let country v_i 's deviation from the strategy profile W be a $1 \times N$ vector $d_i \in \mathbb{R}^N$ such that $W_{i,\bullet} + d_i$ is a valid strategy. The deviation set $D_i(W)$ is the set of all possible deviations of country v_i from the equilibrium strategy profile W . In game $\Gamma = \{\mathcal{V}, R, c, \mathcal{W}, \xi_i, \theta_i, \mathcal{S}, \preceq_i\}$, a strategy profile W is a pure strategy Nash Equilibrium if no unilateral deviation in strategy by any single country is profitable for that country, that is

$$s(W + e_i d_i) \preceq_i s(W), \forall d_i \in D_i(W) \text{ and } i \in \mathcal{V},$$

where e_i as a $N \times 1$ unit vector whose elements are 0 but the i -th coordinate which is 1.

C. Equilibrium Class

As many power allocation strategy profiles can predict the same equilibrium outcome. This motivates a definition of the "equilibrium class". The definition of *equilibrium class* is

$$\bar{W}_i = \{W_i^* : \text{every } W_i^* \in \mathcal{W}^* \text{ such that } s(W_i^*) = s^i, i \in \{1, 2, \dots, 2^N\}\}$$

is both a subset of the pure strategy equilibrium set \mathcal{W}^* and the set of *all* the equilibrium power allocation strategy profiles that predict an equilibrium outcome s^i . The union of the classes is the pure strategy equilibrium set, $\bigcup_{i \in \{1, 2, \dots, 2^N\}} \bar{W}_i = \mathcal{W}^*$. Proposition 2 in Chapter 2 ("Theory of Power Allocation") of [8] proves the equilibrium classes to be convex polytopes.

IV. RESULTS

A. Results I: Equilibrium Existence

This section discusses the first major result of the power allocation game — the pure strategy Nash equilibrium existence for any parametric variation of the game. This result is motivated from two examples respectively from two strands of the game — games in a "Hobbesian environment", where

no alignments exist in the environment, and games in a "non-Hobbesian environment", where alignments exist in the environment.

1) *Example of a Hobbesian Environment: Exploiting Antagonism:* In this Hobbesian environment, the parameters are:

- 1) Countries' Total Power: $p = [p_1, p_2, p_3] = [8, 6, 4]$.
- 2) Countries' Relations: $r_{12} = r_{23} = r_{13} = \text{foe}$. Therefore, all the foe relations make up a complete graph.
- 3) Countries' Preferences: Assume only the choice axioms for their preferences.

This example shows three equilibrium outcomes, each of which has respectively country 1⁹, 2 and 3 as *the only survivor* (and none of the others having survived). These three equilibrium outcomes are [safe, unsafe, unsafe], [unsafe, safe, unsafe], and [unsafe, unsafe, safe].

To summarize the properties of the equilibrium outcomes, each of them has one and only country which has not exhausted its total power in the antagonism with others (e.g., country 1 in outcome I), while the others have.

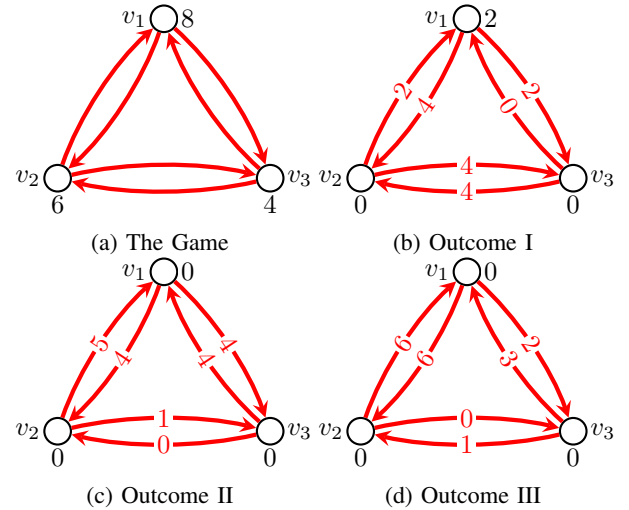


Fig. 3: Unique Survivor in Each Outcome

2) *Example of a Non-Hobbesian Environment: Possible Cycling:* In this non-Hobbesian environment, the parameters are:

- 1) Countries' Total Power: $p = [p_1, p_2, p_3, p_4] = [10, 4, 4, 20]$.
- 2) Countries' Relations: $r_{12} = r_{13} = \text{ally}$, $r_{24} = r_{34} = \text{foe}$ and $r_{23} = r_{14} = \text{none}$.
- 3) Countries' Preferences: Assume for their preferences only the choice axioms.

Parameters-wise, this networked environment is highly non-generic. This especially reflects in the equilibrium outcomes of this power allocation game — equilibrium only occurs when at least one of country 2 and country 3 is in

⁹Country 1 is v_1 in Figure 3a. The same rule applies to the rest of the paper.

the precarious state, such as the outcome shown in Figure 4c. Otherwise, any power allocation profile (e.g., the one in Figure 4b) that does not involve either country 2 or country 3 to be precarious would result in a non-equilibrium cycle—for instance, in Figure 4b, country 1 would divert some resources for supporting country 2 to support country 3. However, when country 3's total support is more than 5, country 4 will divert resources to continue overwhelming country 3. This would result in a cycle, with both country 1 and country 4 constantly updating their allocations. It is worth emphasizing that the kind of cycling scenario *only* happens among a country's allies and foes in games with non-Hobbesian environments.

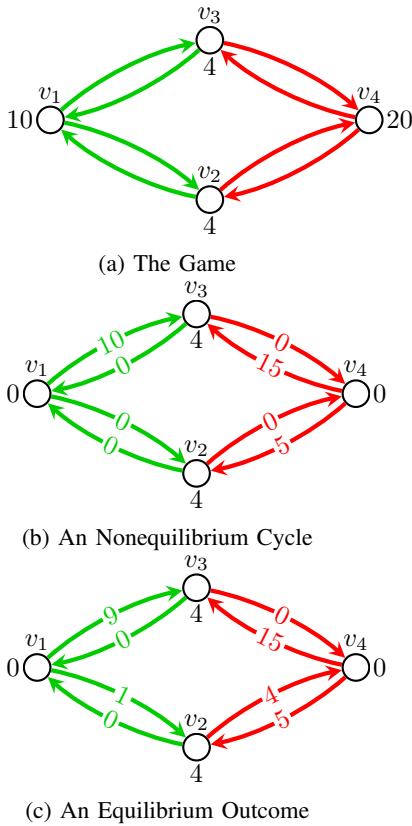


Fig. 4: Nonequilibrium Cycle and Equilibrium

3) *Equilibrium Existence*: Informed by the two examples, one algorithm can be derived to directly construct one equilibrium for any parametric variation of the power allocation game, thus proving the equilibrium existence in Theorem 1 in Chapter 2 (“Theory of Power Allocation”) of [8].

Theorem 1: Assuming the networked international environment, the game $\Gamma = \{\mathcal{V}, R, p, \mathcal{W}, \xi_i, \theta_i, \mathcal{S}, \preceq_i\}$ always has pure strategy Nash Equilibrium.

Several features of the algorithm include:

- 1) firstly, informed by the example of the Hobbesian environment, the algorithm primarily *traverses* each foe relation in the environment consecutively without no particular order and “has” the weaker country exhaust its remaining total power in that foe relation.

- 2) secondly, informed by the non-generic cases of non-Hobbesian environments, countries with positive remaining power after the traversal will keep their foes who have exhausted their total power in their precarious states — this is equivalent to saying that they will not keep offending these foes, as opposed to what country 1 in Figure 3b, country 2 in Figure 3c and country 3 in Figure 3d did. Then the allies of these foes do not need to support them, as they are precarious. Deposit being a corner solution, power allocation strategy profile of this kind is an equilibrium, and may be the only equilibrium in the non-generic cases.
- 3) lastly, the algorithm is only an example of a family of algorithms that can construct a pure strategy Nash equilibrium for *any* parametric variation of the game, and *only* proves equilibrium existence for the game—it never seeks to exhaust the search of *all* pure strategy Nash equilibria.

B. Result II: Unexpected Survival

Besides paving the way for the equilibrium existence result, the previous two examples as just discussed on their own uncovers unintuitive and uniquely interesting dynamics. First of all, the example of Hobbesian game suggests a simple fact—*alignments are not necessary for a country's survival*—the weakest country without any alliances may even be managing to survive through and *only* through the antagonism in the environment, and its survival comes from its ability to exploit others' antagonism in its own favor.

Second, it is obvious from the game in that specific Hobbesian environment that every country can be the unique survivor. This fact will hold as a general result for the game so long as the below parameter conditions hold for the networked international environment :

- 1) Countries' Power: For any country, its total power must be strictly smaller than the total power of all its foes.
- 2) Countries' Relations: Every country is in antagonism with every other country, i.e., their interactions make up a complete graph.
- 3) Countries' Preferences: Countries only need to hold the choice axioms as preferences.

The sufficient conditions for any country to be unique survivor are given below in Proposition 1, which is Proposition 7 in Chapter 5 (“Application I: Balancing”) of [8].

Proposition 1: In a Hobbesian environment $\Gamma = \{\mathcal{V}, R, p, \mathcal{W}, \xi_i, \theta_i, \mathcal{S}, \preceq_i\}$, for any country i , an equilibrium outcome s where $s_i(W) = \text{safe}$, and $j \in \mathcal{V} \setminus \{i\}$ $s_j(W) = \text{unsafe}$ exists if the foe relations make up a complete graph and if

$$p_i < \sum_{v_j \in \Phi_i} p_j, i \in \mathcal{V}$$

Third, when an approximate capacity condition as in Proposition 1 holds for countries with antagonism and all the antagonism make up a complete graph, one particularly interesting equilibrium outcome termed specifically as “balancing

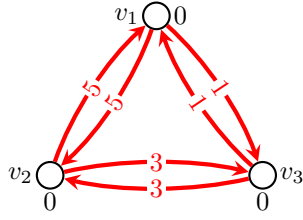


Fig. 5: Balancing Equilibrium

equilibrium” such as the one in Figure 5 exists in games in either Hobbesian or non-Hobbesian environments—in this equilibrium, *every* country with antagonism is in a precarious state. As before, it is extensively discussed in Chapter 5 of [8].

Balancing Equilibrium: A strategy profile W in a power allocation game $\Gamma = \{\mathcal{V}, R, p, \mathcal{W}, \xi_i, \theta_i, \mathcal{S}, \preceq_i\}$ is a *balancing equilibrium* if and only if:

- 1) If $r_{ij} = \text{foe}$, $w_{ii} = w_{jj} = 0$;
- 2) If $r_{ij} = \text{foe}$, $w_{ij} = w_{ji}$;
- 3) If $r_{ij} = \text{ally}$, $w_{ij} = w_{ji} = 0$.

where If $r_{ij} = \text{foe}$, $s_i(W) = s_j(W) = \text{safe}$ (to be exact, precarious).

Proposition 2: In a power allocation game $\Gamma = \{\mathcal{V}, R, p, \mathcal{W}, \xi_i, \theta_i, \mathcal{S}, \preceq_i\}$ where the foe relations make up a complete graph, the game has a balancing equilibrium if and only if

$$p_i \leq \sum_{j \in \Phi_i} p_j, i \in \mathcal{V}$$

C. Results III: Unexpected Vulnerability

This section suggests two examples where countries may not even survive in relatively safe environments, i.e., in an environment where they are supposed to be under protection of allies. These two examples suggest respectively

- 1) A country’s foes may engage in a “divide-and-conquer” strategy—they could still overwhelm this country which is supposed to be well protected by its allies, and this may happen even with these foes not having survived themselves.
- 2) Since a country’s allies may have other allies and foes of their own, this may create weaknesses which are exploitable by the foes of this country.

The examples together suggest that alignments are not sufficient for a country’s survival. Combined with before, this gives us a remarkably simple fact, which is that alignments are *neither necessary nor sufficient* for a country’s survival, and this is the first time it has ever been rigorously proven using a framework of countries’ power allocation.

1) Example of Foes’ Divide and Conquer: The parameters that characterize the environment are:

- 1) Countries’ Total Power: $p = [p_1, p_2, p_3, p_4] = [10, 3, 2, 7]$.
- 2) Countries’ Relations: $r_{12} = r_{23} = \text{ally}$, $r_{14} = r_{24} = r_{34} = \text{foe}$, and $r_{13} = r_{23} = \text{none}$.

- 3) Countries’ Preferences: Assume for countries’ preference the choice axioms.

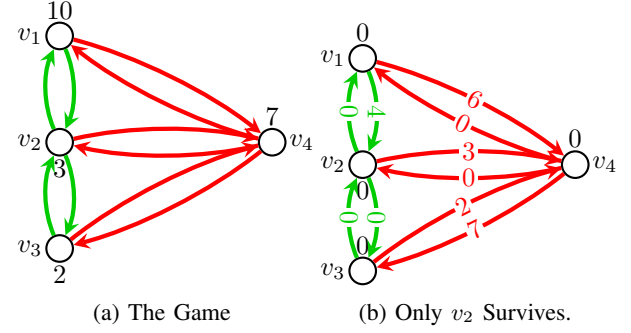


Fig. 6: Unexpected Vulnerability

The two-sided conflict represented on a bipartite foe graph in Figure 7 shows that country 3 is unprotected and unsafe even though the total power of countries 1, 2 and 3 exceed that of their common foe, country 4. Interestingly, the equilibrium outcome in Figure 6b has country 1 make precautions against country 4’s diving and conquering country 2 by preemptively allocating enough support towards country 2.

2) Example of Allies’ Outstanding Obligations: The parameters of the environment are:

- 1) Countries’ Total Power: $p = [p_1, p_2, p_3, p_4, p_5] = [10, 3, 4, 7, 3]$.
- 2) Countries’ Relations: $r_{12} = \text{ally}$, $r_{13} = r_{14} = r_{24} = r_{35} = \text{foe}$, and $r_{15} = r_{23} = r_{34} = r_{45} = \text{none}$.
- 3) Countries’ Preferences: Assume for countries’ preference the choice axioms.

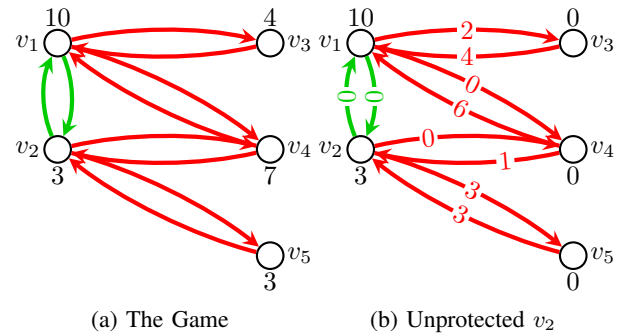


Fig. 7: Ally’s Outside Commitments

This example shows the effects of allies’ outstanding obligations on countries’ survival — specifically, for a country or a group of countries, they may fail to survive even when the total power of itself and its allies exceed that of its foes. In the two-sided conflict in Figure 7, country 2 is not protected by its allies and unsafe. It is important to note that this even happens when $p_1 + p_2 \geq \sum_{v_j \in \Phi_2} p_j$. This example suggests that it is necessary to consider a country’s allies’ own total power and relations in order to understand to what extent the power of the allies are actually available for supporting itself.

There were years in the WWII when the weaker countries (e.g. China), though aligned with stronger alliance members (e.g. the USSR), failed to be protected against the foes (i.e. Japan). This is because while China was at war only with Japan, Russia had both Japan and Germany as enemies. The externalities were transmitted from Germany to China and influenced the latter's survival.

3) *Overcoming the Weaknesses*: The previous two examples combine to imply that a country's survival is heavily influenced by the broader environment this country as well as its neighbors are embedded into. They motivate Proposition 3, which is Proposition 1 in Chapter 5 of [8], specifying one sufficient condition for a group of countries to achieve survival for *every* member.

This condition is that for any subset of countries in this group, their total power must be no smaller than that of their foes, thus guaranteeing the nonexistence of *any* weakness in the group. If this condition does not hold for a proper subset of countries, some foes may exploit this weakness of this group, especially if this proper subset does not have powerful allies.

Proposition 3: Given the power allocation game $\Gamma = \{\mathcal{V}, R, \mathcal{W}, \xi_i, \theta_i, \mathcal{S}, \preceq_i\}$, every country in a group $\mathcal{V}_0 \subset \mathcal{V}$ is safe in equilibrium if for any proper subset of them in \mathcal{V}_0 , their total power is no smaller than that of their foes, formally

$$\sum_{i \in S} p_i \geq \sum_{j \in \Phi_S} p_j, \forall S \subset \mathcal{V},$$

V. DISCUSSION

The power allocation game can be theoretically extended to study a closely related question, which is that how a country can *enhance* the chance of fulfilling its targets. This can only be achieved through *changing the environment*, which lies particularly in changing its own relations or increasing its own power through long-run development strategies or a combination of both. Chapter 3 ("Theory of Relation Dynamics") of [8] has pursued the first possibility theoretically. Two applications of the theory of relation dynamics are developed respectively in Chapter 5 ("Application II: Effectiveness and Stability of Alliances") and Chapter 6 ("Application III: Optimal Network Design") of [8]. And I leave the second possibility for future work.

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